ANTENNAS

Vector and Scalar Potentials

Maxwell's Equations

\[ \nabla \times \mathbf{E} = -j\omega \mathbf{B} \]  \hspace{1cm} (M1)

\[ \nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} \]  \hspace{1cm} (M2)

\[ \nabla \cdot \mathbf{D} = \rho \]  \hspace{1cm} (M3)

\[ \nabla \cdot \mathbf{B} = 0 \]  \hspace{1cm} (M4)

\[ \mathbf{D} = \varepsilon \mathbf{E} \]

\[ \mathbf{B} = \mu \mathbf{H} \]

For a linear, homogeneous, isotropic medium \( \mu \) and \( \varepsilon \) are constant.

Since \( \nabla \cdot \mathbf{B} = 0 \), there exists a vector \( \mathbf{A} \) such that \( \mathbf{B} = \nabla \times \mathbf{A} \) and \( \nabla \cdot (\nabla \times \mathbf{A}) = 0 \). \( \mathbf{A} \) is called the magnetic vector potential. There are infinitely many vectors \( \mathbf{A} \) that satisfy \( \mathbf{B} = \nabla \times \mathbf{A} \) thus we later need to specify \( \nabla \cdot \mathbf{A} \).

(M1) can be written as

\[ \nabla \times \mathbf{E} = -j\omega \nabla \times \mathbf{A} \Rightarrow \nabla \times (\mathbf{E} + j\omega \mathbf{A}) = 0 \]  \hspace{1cm} (1)

\[ \nabla \times \text{Vector} = 0 \Rightarrow \text{Vector} = \nabla (\text{some scalar}) \]  \hspace{1cm} (2)

\[ \mathbf{E} + j\omega \mathbf{A} = -\nabla \varphi \quad \varphi : \text{scalar potential} \]  \hspace{1cm} (3)

\[ \mathbf{E} = -j\omega \mathbf{A} - \nabla \varphi \]  \hspace{1cm} (4)

(M2) can be written as

\[ \nabla \times \mathbf{B} = \mu \mathbf{J} + j\omega \varepsilon \mathbf{E} \]  \hspace{1cm} (5)

\[ \nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} + j\omega \varepsilon (\mu \mathbf{J} - \nabla \varphi) \]  \hspace{1cm} (6)

\[ -\nabla^2 \mathbf{A} + \nabla (\nabla \cdot \mathbf{A}) = \mu \mathbf{J} + \omega^2 \mu \varepsilon \mathbf{A} - j\omega \epsilon \mu \nabla \varphi \]  \hspace{1cm} (7)
To simplify the equation, choose $\nabla \cdot A$ as

$$\nabla \cdot A = -j\omega \mu \varepsilon \phi$$ \hspace{1cm} (8)

This is called the **Lorentz Condition**. This leads to

$$\nabla^2 A + \omega^2 \mu \varepsilon A = -\mu J$$ \hspace{1cm} \textit{D'Alembert's Equation} \hspace{1cm} (9)

The problem now consists of finding the vector potential $A$ due to a source $J$. From the knowledge of $A$, $E$ and $H$ can be determined.

Review spherical coordinates, gradient, divergence, curl, and laplacian in spherical coordinates. (Textbook, Appendix A pp. 689-691).

In spherical coordinates, the solution for the vector potential $A(r)$ is given by

$$A(r) = \mu \int \int \int_{V'} \frac{J(r') e^{-j|\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|} \, dv'$$ \hspace{1cm} (10)

In the above equation, $V'$ is the volume defining the source over which the integration is performed. $\mathbf{r}$ is the vector from the origin of the coordinate system, $O$, to the observer and $\mathbf{r}'$ is the vector from $O$ to a source point within $V'$.

![Figure 1](image)

From $A$, use Maxwell's Equations to derive $E$ and $H$.

$$\mathbf{B} = \nabla \times \mathbf{A}$$ \hspace{1cm} (11)

$$j\omega \varepsilon \mathbf{E} = \nabla \times \mathbf{H}$$ \hspace{1cm} (12)
Hertzian Dipole

Assume a source point infinitely small in size located at the origin of the coordinate system with elemental length $dl$, and driven by a current with strength $I_0$ in the $+z$ direction. The equation for the current density of such a system is given by

$$\mathbf{J}(r') = iz \frac{I_0 dl}{4\pi} \delta(x')\delta(y')\delta(z') \quad (13)$$

Upon substitution in (10)

$$\mathbf{A}(r) = iz \frac{\mu I_0 dl}{4\pi r} \exp(-j\beta r) \quad (14)$$

In spherical coordinates: $iz = ir\cos\theta - i\theta\sin\theta$

$$\mathbf{A}(r) = (ir\cos\theta - i\theta\sin\theta) \frac{\mu I_0 dl}{4\pi r} \exp(-j\beta r) \quad (15)$$

$$A_r = \frac{\mu}{4\pi r} I_0 dl \cos\theta \exp(-j\beta r) \quad (16)$$

$$A_\theta = -\frac{\mu}{4\pi r} I_0 dl \sin\theta \exp(-j\beta r) \quad (17)$$

Calculate $E$ and $H$ fields.

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad (18)$$

$$H_r = \frac{1}{r \sin\theta} \left[ \frac{\partial}{\partial \theta} (A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi} \right] = 0 \quad (19)$$

$$H_\theta = \frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (rA_\phi) \right] = 0 \quad (20)$$

$$H_\phi = \frac{1}{r} \left[ \frac{\partial}{\partial r} (rA_\theta) - \frac{\partial A_\phi}{\partial \theta} \right] \quad (21)$$

$$H_\phi = \frac{I_0 dl}{4\pi r} \sin\theta \exp(-j\beta r) j\beta(1 + \frac{1}{j\beta r}) \quad (22)$$

Using $\mathbf{E} = \frac{1}{j\omega\varepsilon} \nabla \times \mathbf{H}$, we can derive the components of the electric field.
\[ E_r = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (H_\phi \sin \theta) \right] \frac{1}{j \omega \epsilon} \]  

(23)

\[ E_r = \frac{2 \cos \theta I_1 \, dl}{4 \pi r^2} \left\{ e^{-j \beta r} \right\} j \beta \left( 1 + \frac{1}{j \beta r} \right) \frac{1}{j \omega \epsilon} \]  

(24)

\[ E_\theta = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r H_\phi) \right] \frac{1}{j \omega \epsilon} \]  

(25)

\[ E_\theta = -\frac{j \beta I_1 \, dl}{4 \pi r} \sin \theta \left\{ e^{-j \beta r} \right\} \left[ -j \beta \left( 1 + \frac{1}{j \beta r} - \frac{1}{\beta^2 r^2} \right) \right] \frac{1}{j \omega \epsilon} \]  

(26)

In summary, the exact solutions for the fields of an infinitesimal antenna are given by

\[ E_r = \frac{j \beta I_1 \, dl}{4 \pi r} e^{-j \beta r} 2 \cos \theta \left( \frac{1}{j \beta r} + \frac{1}{(j \beta r)^2} \right) \sqrt{\frac{\mu}{\epsilon}} \]  

(27)

\[ E_\theta = \frac{j \beta I_1 \, dl}{4 \pi r} e^{-j \beta r} \sin \theta \left( 1 + \frac{1}{j \beta r} + \frac{1}{(j \beta r)^2} \right) \sqrt{\frac{\mu}{\epsilon}} \]  

(28)

\[ E_\phi = 0 \]  

(29)

\[ H_\phi = \frac{j \beta I_1 \, dl}{4 \pi r} e^{-j \beta r} \sin \theta \left( 1 + \frac{1}{j \beta r} \right) \]  

(30)

\[ H_r = H_\theta = 0 \]  

In most practical cases, the observer is located several wavelengths away from the source. This defines a \textit{far field} region which is the region where the distance from the source to the observer is much larger than the wavelength \( \lambda = \frac{2 \pi}{\beta} \). In this case \( r >> \lambda \) so that \( \beta r >> 1 \); consequently, the terms varying as \( 1/r^2 \) and \( 1/r^3 \) can be neglected. The far-field solutions for the infinitely small antenna thus become

\[ E_r = 0 \]  

(30)

\[ E_\theta = \sqrt{\frac{\mu}{\epsilon}} \frac{j \beta I_1 \, dl}{4 \pi r} e^{-j \beta r} \sin \theta \]  

(31)

\[ H_\phi = \frac{j \beta I_1 \, dl}{4 \pi r} e^{-j \beta r} \sin \theta \]  

(32)
Note that the ratio $E_\theta/H_\phi$ is the characteristic impedance $\eta$ of the propagation medium. Over a small region, the far field solution is a plane-wave solution since the electric and magnetic fields are in phase, perpendicular to each other and their ratio is the intrinsic impedance $\eta$, and are perpendicular to the direction of propagation. However, unlike plane waves, the far field solution is a function of the elevation angle $\theta$, and does not have constant magnitude ($1/r$ dependence).

**Radiation Patterns**

The graph that describes the far-field strength versus the elevation angle at a fixed distance is called the *radiation pattern* of the antenna. In general, radiation patterns vary with $\theta$ and $\phi$. The distance from the dipole to a point on the radiation pattern is proportional to the field intensity or power density observed in that direction. Figure 2 shows the E-field and power density radiation patterns of a Hertzian dipole. As can be verified these patterns are based on the $\sin \theta$ and $\sin^2 \theta$ dependence of the E-field and power density respectively.

![ Radiation Pattern Diagrams](image)

Figure 2. (a) Radiation pattern for E field (b) Radiation pattern for power density

**Time Average Power in Radiation Zone**

In order to calculate the power radiated in the far field, we need to determine the time-average Poynting vector or power density $<P>$.

$$< P > = \frac{1}{2} \text{Real} \left[ E \times H^* \right] = \frac{i}{2} \text{Real} \left[ \frac{H}{\sqrt{\varepsilon}} |H_\phi|^2 \right]$$  

(33)
\[ <\mathbf{P}> = \frac{i_\eta}{2} \text{Real} \left[ \frac{\beta I_0 dl}{4\pi r} \right] \sin^2 \theta \]  

(34)

The total power radiated \( P_T \) at a distance \( r \) is by definition obtained by integrating the Poynting vector over a sphere of radius \( r \).

\[
\text{Total Power} = P_T = \int_0^{2\pi} \int_0^\pi <\mathbf{P}> \cdot d\mathbf{S}
\]

(35)

d\(S\) is the elemental surface of radius \( r \) and is given by

\[
d\mathbf{S} = i_r r^2 \sin \theta \, d\theta \, d\phi
\]

(36)

so that

\[
P_T = \int_0^{2\pi} \int_0^\pi \frac{i_\eta}{2} \left( \frac{\beta I_0 dl}{4\pi} \right)^2 r^2 \sin^3 \theta \, d\theta \, d\phi
\]

(37)

\[
P_T = \frac{\eta}{2} \left( \frac{\beta I_0 dl}{4\pi} \right)^2 \int_0^{\pi} \sin^3 \theta \, d\theta
\]

(38)

\[
P_T = \frac{4\pi \eta}{3} \left( \frac{\beta I_0 dl}{4\pi} \right)^2 = \pi \frac{\eta}{3} \frac{\|dl\|^2}{\lambda} \|I_0\|^2
\]

(39)

The **directive gain** is a figure of merit defined as

\[
\text{Directive Gain} = \frac{\text{Poynting power density}}{\text{Average Poynting power density over area of sphere of radius } r}
\]

or

\[
\text{Directive Gain} = \frac{<\mathbf{P}>}{P_T/4\pi r^2}
\]

(40)

For an infinitesimal antenna, we get

\[
\text{Directive Gain} = \frac{\eta}{2} \left( \frac{\beta I_0 dl}{4\pi} \right)^2 \frac{\sin^2 \theta}{4\pi r^2} = \frac{3}{2} \sin^2 \theta
\]

(41)
The *directivity* is the directive gain in the direction of its maximum value. For an infinitesimal antenna, the direction of maximum value is for $\theta = \pi/2$ and the directivity is 1.5.

**Radiation Resistance**

The *radiation resistance* of an antenna is defined as the value of the resistor that would dissipate an equal amount of power than the power radiated for the same value of current. Using

$$P_T = \frac{1}{2} R_{rad} I_o^2$$

(42)

We get

$$R_{rad} = \frac{2P_T}{I_o^2}$$

(43)

For an infinitesimal antenna, we get

$$R_{rad} = \frac{2}{I_o^2} \left( \frac{4\pi}{3} \right) \eta \left| \frac{\beta I_o dl}{4\pi} \right|^2$$

(44)

Using $\beta = \frac{2\pi}{\lambda}$, and $\eta = 120\pi$ ohms, we obtain

$$R_{rad} = 80\pi^2 \left( \frac{dl}{\lambda} \right)^2 \Omega$$

(45)