WAVEGUIDES

Maxwell's Equation

\[ \nabla^2 E + \omega^2 \mu \varepsilon E = 0 \quad \text{(A)} \]

\[ \nabla^2 H + \omega^2 \mu \varepsilon H = 0 \quad \text{(B)} \]

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set \( E_z = 0 \). This defines the TE modes.

From \( \nabla \times E = -\mu \frac{\partial H}{\partial t} \), we have

\[
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \Rightarrow +j\beta_z E_y = -j\omega \mu H_x
\]  

\[
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \Rightarrow -j\beta_z E_x = -j\omega \mu H_y
\]  

\[
\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega \mu H_z
\]

From \( \nabla \times E = -j\omega \mu H \), we get

\[
j\omega \varepsilon E = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
H_x & H_y & H_z
\end{vmatrix}
\]

\[
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega \varepsilon E_x \Rightarrow \frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega \varepsilon E_x
\]
\[
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \Rightarrow -j\beta_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \tag{5}
\]

\[
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \tag{6}
\]

We want to express all quantities in terms of \(H_z\).

From (2), we have

\[
H_y = \frac{\beta_z E_x}{\omega\mu} \tag{7}
\]

in (4)

\[
\frac{\partial H_z}{\partial y} + j\beta_z^2 \frac{E_x}{\omega\mu} = j\omega\varepsilon E_x \tag{8}
\]

Solving for \(E_x\)

\[
E_x = \frac{j\omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial y} \tag{9}
\]

From (1)

\[
H_x = \frac{-\beta_z E_y}{\omega\mu} \tag{10}
\]

in (5)

\[
+j \frac{\beta_z^2 E_y}{\omega\mu} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \tag{11}
\]

so that

\[
E_y = \frac{-j\omega\mu}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial x} \tag{12}
\]

\[
H_x = \frac{j\beta_z}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial x} \tag{13}
\]

\[
H_y = \frac{j\beta_z}{\beta_z^2 - \omega^2 \mu\varepsilon} \frac{\partial H_z}{\partial y} \tag{14}
\]
\[ E_z = 0 \]  

Combining solutions for \( E_x \) and \( E_y \) into (3) gives

\[
\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = [\beta_z^2 - \omega^2 \mu \varepsilon] H_z \]  

**RECTANGULAR WAVEGUIDES**

If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

**TE Modes**

The general solution for TE modes with \( E_z = 0 \) is obtained from (16)

\[
H_z = e^{-j\beta_x z} \left[ Ae^{-j\beta_y x} + Be^{+j\beta_y x} \right] \left[ Ce^{-j\beta_y y} + De^{+j\beta_y y} \right] \]  

(17)

\[
E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} e^{-j\beta_x z} \left[ Ae^{-j\beta_y x} + Be^{+j\beta_y x} \right] \left[ Ce^{-j\beta_y y} + De^{+j\beta_y y} \right] \]  

(18)

\[
E_x = \frac{j\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} e^{-j\beta_x z} \left[ Ae^{-j\beta_y x} + Be^{+j\beta_y x} \right] \left[ Ce^{-j\beta_y y} + De^{+j\beta_y y} \right] \]  

(19)

At \( y = 0 \), \( E_x = 0 \) which leads to \( C = D \)

At \( x = 0 \) \( E_y = 0 \) which leads to \( A = B \)

\[
H_z = H_0 e^{-j\beta_z z} \cos \beta_x x \cos \beta_y y \]  

(20)

\[
E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z} \sin \beta_x x \cos \beta_y y \]  

(21)
$E_x = \frac{j\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z} \cos \beta_x x \sin \beta_y y$  \hspace{1cm} (22)

At $x=a$, $E_y=0$; this leads to $\beta_x = \frac{m \pi}{a}$

At $y=b$, $E_x=0$; this leads to $\beta_y = \frac{n \pi}{b}$

The dispersion relation is obtained by placing (20) in (16)

$\beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2 \mu \varepsilon$  \hspace{1cm} (23)

$\left(\frac{m \pi}{a}\right)^2 + \left(\frac{n \pi}{b}\right)^2 + \beta_z^2 = \omega^2 \mu \varepsilon$  \hspace{1cm} (24)

and

$\beta_z = \sqrt{\omega^2 \mu \varepsilon - \left(\frac{m \pi}{a}\right)^2 - \left(\frac{n \pi}{b}\right)^2}$  \hspace{1cm} (25)

The guidance condition is

$\omega^2 \mu \varepsilon > \left(\frac{m \pi}{a}\right)^2 + \left(\frac{n \pi}{b}\right)^2$  \hspace{1cm} (26)

or

$f > f_c$ where $f_c$ is the cutoff frequency of the TE$_{mn}$ mode given by the relation

$f_c = \frac{1}{2\sqrt{\mu \varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$  \hspace{1cm} (27)

The TE$_{mn}$ mode will not propagate unless $f$ is greater than $f_c$. Obviously, different modes will have different cutoff frequencies.

**TM Modes**

The transverse magnetic modes for a general waveguide are obtained by assuming $H_z = 0$. By duality with the TE modes, we have
\[ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = [\beta z^2 - \omega^2 \mu \varepsilon] E_z \] (28)

with general solution

\[ E_z = e^{-j\beta_z z} \left[ A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] C e^{-j\beta_y y} + D e^{+j\beta_y y} \] (29)

The boundary conditions are

At \( x=0 \), \( E_z=0 \) which leads to \( A=-B \)

At \( y=0 \), \( E_z=0 \) which leads to \( C=-D \)

At \( x=a \), \( E_z=0 \) which leads to \( \beta_x = \frac{m\pi}{a} \)

At \( y=b \), \( E_z=0 \) which leads to \( \beta_y = \frac{n\pi}{b} \)

so that the generating equation for the \( \text{TM}_{mn} \) modes is

\[ E_z = E_o e^{-j\beta_z z} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \] (30)

**NOTE:** THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR \( \text{TE} \) AND \( \text{TM} \) MODES.

For additional information on the field equations see Rao (6th Edition), page 607 Table 9.1.

There is no \( \text{TE}_{00} \) mode

There are no \( \text{TM}_{m0} \) or \( \text{TM}_{0n} \) modes

The first \( \text{TE} \) mode is the \( \text{TE}_{10} \) mode

The first \( \text{TM} \) mode is the \( \text{TM}_{11} \) mode

**Impedance of a Waveguide**

For a \( \text{TE} \) mode, we define the transverse impedance as
\[ \eta_{gTE} = \frac{-E_Y}{H_x} = \frac{E_x}{H_y} = \frac{\omega \mu}{\beta_z} \]

From the relationship for \( \beta_z \) and using

\[ f_c^2 = \frac{1}{4 \pi^2 \mu \varepsilon} \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right] \]

we get

\[ \eta_{gTE} = \eta \sqrt{1 - \frac{f_c^2}{f^2}} \]  \hspace{1cm} (31)

where \( \eta \) is the intrinsic impedance \( \eta = \sqrt{\frac{\mu}{\varepsilon}} \). Analogously, for TM modes, it can be shown that

\[ \eta_{gTM} = \eta \sqrt{1 - \frac{f_c^2}{f^2}} \]  \hspace{1cm} (33)

**Power Flow in a Rectangular Waveguide (TE10)**

The time-average Poynting vector for the TE\(_{10} \) mode in a rectangular waveguide is given by

\[ \langle \mathbf{P} \rangle = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} = \hat{z} \frac{|E_0|^2}{2} \frac{\beta_z \sin^2 \frac{\pi x}{a}}{\omega \mu} \]  \hspace{1cm} (34)

\[ \langle \text{Power} \rangle = \int_0^a \int_0^b \frac{|E_0|^2}{2} \frac{\beta_z \sin^2 \frac{\pi x}{a}}{\omega \mu} \ dxdy \]  \hspace{1cm} (35)

\[ \langle \text{Power} \rangle = \frac{|E_0|^2 \beta_z ab}{4 \omega \mu} \frac{\beta_z \sin^2 \frac{\pi x}{a}}{\omega \mu} = \frac{|E_0|^2 \beta_z ab}{4 \eta_{gTE10}} \]  \hspace{1cm} (36)

Therefore the time-average power flow in a waveguide is proportional to its cross-section area.