1. Consider the time-harmonic charge distribution

\[ \rho (\mathbf{r}, t) = q \left[ \delta (\mathbf{r} - h\hat{x}) - \delta (\mathbf{r} + h\hat{x}) + \delta (\mathbf{r} - h\hat{y}) - \delta (\mathbf{r} + h\hat{y}) \right] \cos(\omega t) \frac{C}{m^3} \]

in free space.

a) Write down the phasors for the charge density \( \rho(\mathbf{r}, t) \) and its time derivative \( \frac{\partial \rho}{\partial t} \).

b) Calculate the divergence \( \nabla \cdot \mathbf{J} \) of the current density associated with \( \rho(\mathbf{r}, t) \) above. **Hint:** continuity equation.

c) What is the retarded scalar potential phasor \( \tilde{V} \) generated by \( \rho(\mathbf{r}, t) \) above?

2. Give the time-harmonic expressions with the frequency \( \omega \) for the fields \( \mathbf{A}(\mathbf{r}, t) \) with the following phasors:

a) \( \tilde{\mathbf{A}}(\mathbf{r}) = j2\hat{x} - 2\hat{y} \)

b) \( \tilde{\mathbf{A}}(\mathbf{r}) = \hat{x} + j\omega\hat{y} + 2\omega^2\hat{z} = \hat{x} + j\omega\hat{y} - 2(j\omega)^2\hat{z} \)

c) \( \tilde{\mathbf{A}}(\mathbf{r}) = (j\hat{x} + 2\hat{y})e^{-jkz} \) where \( k \) is a constant.

3. Consider the time-harmonic vector field

\[ \mathbf{A}(\mathbf{r}, t) = \hat{x}A_x(\mathbf{r}) \cos(\omega t + \phi_x(\mathbf{r})) + \hat{y}A_y(\mathbf{r}) \cos(\omega t + \phi_y(\mathbf{r})) + \hat{z}A_z(\mathbf{r}) \cos(\omega t + \phi_z(\mathbf{r})) \]

where \( A_x(\mathbf{r}), \phi_x(\mathbf{r}), \) etc. are real valued functions of space variables \( \mathbf{r} \equiv (x, y, z) \).

a) Show that its complex valued phasor \( \tilde{\mathbf{A}}(\mathbf{r}) \) can be written in the form

\[ \tilde{\mathbf{A}}(\mathbf{r}) = \mathbf{A}_R(\mathbf{r}) + j\mathbf{A}_I(\mathbf{r}), \]

where \( \mathbf{A}_R(\mathbf{r}) \) and \( \mathbf{A}_I(\mathbf{r}) \) are real vectors and identify \( \mathbf{A}_R(\mathbf{r}) \) and \( \mathbf{A}_I(\mathbf{r}) \).

b) Show that the time average of the cross product \( \mathbf{A}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \) of two time-harmonic vector fields \( \mathbf{A}(\mathbf{r}, t) \) and \( \mathbf{B}(\mathbf{r}, t) \) of the same period equals

\[ \frac{1}{2} \Re \left\{ \tilde{\mathbf{A}}(\mathbf{r}) \times \left( \tilde{\mathbf{B}}(\mathbf{r}) \right)^* \right\} \]

in terms of the corresponding phasors.

4.

a) Find the curl and divergence for each of the following vectors in spherical coordinates:

i. \( \mathbf{A} = r^2\hat{\mathbf{r}} + r \sin \theta \hat{\mathbf{\phi}} \),

ii. \( \mathbf{B} = \frac{e^{-r}}{r} \hat{\mathbf{\phi}} \).

b) Find the gradient for each of the scalar functions in spherical coordinates:

i. \( U = \frac{\sin \theta}{r} \),

ii. \( V = r \cos \theta \).

5.
a) The Laplacian $\nabla^2 V$ of a scalar $V = V(x, y, z)$ is, by definition, the divergence $\nabla \cdot (\nabla V)$ of the gradient $\nabla V$ of the scalar field $V$. Also, 

$$\nabla \cdot (f \mathbf{A}) = f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

for all (differentiable) scalar and vector fields $f = f(x, y, z)$ and $\mathbf{A} = \mathbf{A}(x, y, z)$, respectively.

Given the above facts, show that

$$\nabla^2 \frac{e^{-jk|r|}}{4\pi\varepsilon_0 r} + 2\nabla \left( \frac{1}{4\pi\varepsilon_0 r} \right) \cdot \nabla \left( e^{-jk|r|} \right) + \nabla \cdot \nabla \left( \frac{1}{4\pi\varepsilon_0 r} \right) e^{-jk|r|},$$

where $k$ is some constant, $\mathbf{r} \equiv (x, y, z) = \hat{x} x + \hat{y} y + \hat{z} z$, and $|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$.

b) Show that the first two terms of on the right in (a) can be simplified as

$$\nabla^2 \frac{e^{-jk r}}{4\pi\varepsilon_0 r} + 2\nabla \left( \frac{1}{4\pi\varepsilon_0 r} \right) \cdot \nabla \left( e^{-jk r} \right) = -k^2 \frac{e^{-jk r}}{4\pi\varepsilon_0 r}.$$

6. Alternative to (5).

Using the definition of the Laplacian in spherical coordinates (see Example 3 in Lecture 5), show that

$$\nabla^2 \left( \frac{f(r)}{r} \right) = \frac{1}{r} \frac{\partial^2 f}{\partial r^2}$$

for an arbitrary function $f(r)$ of radial distance $r$ from the origin of a spherical coordinate system. This result was used in Lecture 4 in the proof of the retarded potential solution of the forced wave equation for the scalar potential $\tilde{V}$ in phasor form.