1. There exist a number ways to approach this problem. The more conceptual approach is presented here which is based on the fundamental principles involving the dipole element. At the end, the more rigorous, mathematical approach is highlighted as a confirmation.

   a) Consider the \( \hat{z} \)-directed Hertzian dipole shown in the figure below. It is very important to understand and visualize what the general far field \( \mathbf{E} \) expression indicates.

   Consider the \( xz \) plane as an example.
   
   - The expression indicates the E-field strength varies as \( \sin \theta \), hence the figure-eight like pattern. Since the field strength is not a function of \( \phi \), the pattern is axis-symmetric about \( z \) which creates the doughnut/bagel shape pattern in three-dimensions (for any angle \( \phi \)). The main points of interest are well known: the dipole does not radiate along its axis (\( \theta = 0^\circ \) in this case) and its maximum is along the horizon (in the \( xy \) plane, \( \theta = 90^\circ \)).
   - The figure also details the \textit{vector} properties of the expression. The dipole is \( \hat{\theta} \) polarized, the arrows along the larger arc gives a generalization to what direction the \( \hat{\theta} \) unit vector points. Recall they exist over all space and the arrows on along the field straight pattern gives some idea what direction they point along the pattern. The take away point: in the \( xy \) plane, the plane perpendicular to the dipole axis, \( \hat{\theta} = -\hat{z} \) (the direction opposite the dipole orientation).

With that insight, it should not be hard to understand what happens when the dipole is oriented along the \( x \) or \( y \) direction or even tilted.

   i. Consider the point \((x, y, z) = (d, 0, 0)\) in the figure below, it implies \( r = d \) and the observation point is along \( \theta = 90^\circ \) and \( \phi = 0^\circ \).
Given the previous discussion it should be easy to see the following:

- The $\hat{x}$-directed dipole does not contribute to the field expression at this point, the point is in the null of the pattern.
- The $\hat{y}$-directed dipole contributes a component in the $-\hat{y}$ direction (the $xz$-plane is in a perpendicular to the dipole’s axis). Furthermore, it is easy to see that $-\hat{y} = -\hat{\phi}$ at this point.
- And of course for the $\hat{z}$-directed dipole, it contributes a $\hat{\theta} = -\hat{z}$ component.

Since the dipoles have the same dipole moment and are co-located at the origin, the far field expression then becomes a superposition of each element’s contribution, i.e.,

$$\tilde{E}(r = d\hat{x}) = \eta_0 j k I \Delta \ell e^{-jkd} (\hat{\theta} - \hat{\phi}) \frac{V}{m} = \eta_0 j k I \Delta \ell e^{-jkd} (-\hat{y} - \hat{z}) \frac{V}{m}.$$  

One can see that the magnitude is $\sqrt{2}$ times that of a single dipole element at this point.

ii. At $(x, y, z) = (d, d, d)$, which implies $r = \sqrt{d^2 + d^2 + d^2} = d\sqrt{3}$, it may not be as easy to visualize the directional components in terms of $\hat{\theta}$ and $\hat{\phi}$ at the observation point. A simpler method would be to utilize the principal of superposition on the dipole elements themselves. The dipoles are equally weighted, i.e., have the same dipole moment and they form a triad. Therefore, construct an equivalent single dipole element out of the three; the result is analogous to a vector superposition of the three scaled unit vector along $\hat{x}$, $\hat{y}$, and $\hat{z}$. Graphically, the image in the figure below of the left equates to the image on the right.
Of course the the dipole moment of the equivalent dipole would be $\sqrt{3}$ times larger than the the individual dipole moment. Nevertheless, the key is to note that the equivalent dipole is oriented along the $\hat{r} = \frac{1}{\sqrt{3}} (\hat{x} + \hat{y} + \hat{z})$ unit vector direction. This is along the same direction our observation point, which is in the null of the equivalent dipole's radiation pattern. Hence at this observation point the component contributions from each of the dipole elements cancel. Therefore,

$$\mathbf{E}(r = d\hat{x} + d\hat{y} + d\hat{z}) = 0 \text{ V/m.}$$

b) With these results its easy to see that NO, the three-antenna system is not an isotropic radiator. Recall, an isotropic radiator, for example, at a distance $d$ away from the antenna radiates an equal field strength (power) in all directions. Based on the results in a), it is clear that the field strength of $\tilde{\mathbf{E}}$ varies spatially, i.e., it’s a function of position. As a matter of fact, the pattern over all space will be that same doughnut/bagel shape pattern in three-dimensions-only in this case is symmetric about the equivalent dipole’s axis!

c) Bonus: Mathematical derivation for the interested reader

When the far-field is of interest, it should not be surprising that the magnetic vector potential can be approximated as

$$\tilde{\mathbf{A}}(r) = \frac{e^{-jk\hat{r} \cdot r}}{r} \left[ \frac{\mu_0}{4\pi} \int \tilde{\mathbf{J}}(r') e^{jk\hat{r} \cdot r'} d^3r' \right],$$

this will be discussed more in problem 4. Nevertheless the relationship serves as a means to transform or translate the source contributions to the far field. Given that we can write

$$\tilde{\mathbf{E}}(r) = \frac{1}{j\omega\epsilon_0} \left( \nabla \times \nabla \times \tilde{\mathbf{A}} \right),$$

which in the far field becomes equivalent to

$$\tilde{\mathbf{E}}(r) = -j\omega\mu_0 \left( A_\theta(r) \hat{\theta} + A_\phi(r) \hat{\phi} \right) \quad (1)$$

by approximating $\nabla \to -jk\hat{r}$. The result indicate the electric far field is given by the component of $-j\omega\mu_0\tilde{\mathbf{A}}(r)$ that is transverse to $\hat{r}$.

Then in light of these results, one can sum of the contributions of the vector potential for the $\hat{x}$-, $\hat{y}$-, and $\hat{z}$-directed Hertzian dipoles as

$$\tilde{\mathbf{A}}(r) \bigg|_{total} = \tilde{\mathbf{A}}(r) \bigg|_{\hat{x}\text{-directed}} + \tilde{\mathbf{A}}(r) \bigg|_{\hat{y}\text{-directed}} + \tilde{\mathbf{A}}(r) \bigg|_{\hat{z}\text{-directed}} = \frac{\mu_0}{4\pi} I\Delta\ell \frac{e^{-jkr}}{r} [\hat{x} + \hat{y} + \hat{z}].$$

In order to utilize this expression convert from rectangular to spherical coordinates, i.e., make a coordinate transformation with the vector components and ignore the radial components. Doing so then applying (1), one can show that the total radiation field of the three Hertzian dipole system amounts to

$$\tilde{\mathbf{E}}(r) = \eta_0 jk I\Delta\ell \frac{e^{-jkr}}{4\pi r} \left\{ (-\cos \theta \cos \phi - \cos \theta \sin \phi + \sin \theta) \hat{\theta} + (\sin \phi - \cos \phi) \hat{\phi} \right\} \text{ V/m.} \quad (2)$$

The result verifies with parts i) and ii) above:
\begin{itemize}
  \item \((x, y, z) = (d, 0, 0) \rightarrow \theta = 90^\circ\) and \(\phi = 0^\circ\). Then in (2),
    \[ \mathbf{\tilde{E}}(r) = \eta_0 j k I \Delta\ell \frac{e^{-jkd}}{4\pi d} \left( \hat{\theta} - \hat{\phi} \right) V/m. \]
  \item \((x, y, z) = (d, d, d) \rightarrow \theta = \tan^{-1}\left( \frac{\sqrt{x^2 + y^2}}{z} \right) = 54.7^\circ\) \(\text{and} \ \phi = 45^\circ\). Then in (2),
    \[ \mathbf{\tilde{E}}(r) = 0 \]
\end{itemize}

2. For the \(\hat{z}\)-directed half-wave length dipole,
\[ \ell(\theta) = \frac{\lambda \cos\left( \frac{\pi}{2} \cos \theta \right)}{\sin^2 \theta}, \]
evaluating \(\ell(\theta = 0) = \frac{0}{0} \rightarrow \) an indeterminate form. Use l’Hospital’s rule as indicated.

a) \[
\ell(0) = \frac{\lambda}{\pi} \lim_{\theta \to 0} \frac{-\sin\left( \frac{\pi}{2} \cos \theta \right) \left( -\frac{\pi}{2} \sin \theta \right)}{2 \sin \theta \cos \theta} = \frac{\lambda}{\pi} \lim_{\theta \to 0} \frac{-\sin\left( \frac{\pi}{2} \cos \theta \right) \left( -\frac{\pi}{2} \right)}{2 \cos \theta} = \frac{\lambda}{\pi} \frac{\pi}{4} = \frac{\lambda}{4}.
\]
The result indicates that the effective length of the half-wave length dipole looks only a quarter-wave length long from \(\theta = 0^\circ\).

b) The plot of \(\ell(\theta)\) for \(0^\circ \leq \theta \leq 180^\circ\), is shown below.
\[
\frac{\ell(\theta)}{\lambda} = \frac{1}{\pi} \frac{\cos\left( \frac{\pi}{2} \cos \theta \right)}{\sin^2 \theta} \ \text{for} \ -180^\circ \leq \theta \leq 180^\circ
\]
c) The plot of $\ell(\theta) \sin \theta$ for $0^\circ \leq \theta \leq 180^\circ$ is given below.

$$\left| \frac{\ell(\theta)}{\lambda} \sin \theta \right| = \left| \frac{1}{\pi} \frac{\cos \left( \frac{\theta}{2} \cos \theta \right)}{\sin^2 \theta} \sin \theta \right|$$

for $-180^\circ \leq \theta \leq 180^\circ$

d) We know that
\[ \frac{E}{\lambda_{\text{dipole}}} = \hat{\theta} j \eta I k \ell(\theta) \sin \theta \frac{e^{-jkr}}{4\pi r} \]
\[ = \hat{\theta} 2j \eta I \cos \left( \frac{\pi}{2} \cos \theta \right) \frac{e^{-jkr}}{4\pi r} , \]

and

\[ \frac{H}{\lambda_{\text{dipole}}} = \hat{\phi} \frac{\lambda_{\text{dipole}}}{\eta} \]
\[ = \hat{\phi} 2j I \cos \left( \frac{\pi}{2} \cos \theta \right) \frac{e^{-jkr}}{4\pi r} \]

e) The time-average power density for the half-wave dipole is

\[ \langle |E \times H| \rangle = \frac{|E|^2}{2\eta} \]
\[ = \frac{2\eta I \cos \left( \frac{\pi}{2} \cos \theta \right) \frac{e^{-jkr}}{4\pi r}^2}{2\eta} \]
\[ = \frac{\eta I^2 \cos \left( \frac{\pi}{2} \cos \theta \right)^2}{8\pi^2 r^2} \frac{\left| \cos \left( \frac{\pi}{2} \cos \theta \right) \sin \theta \right|^2}{8\pi^2 r^2} , \]

and the plot for \( \frac{|E \times H|}{\eta I} = \frac{1}{8\pi^2} \left| \cos \left( \frac{\pi}{2} \cos \theta \right) \frac{\sin \theta}{\sin \theta} \right|^2 \) is as below (see next page).
3. For the sphere centered at \((x, y, z) = (0, 0, r)\), define \(\alpha\) as the half subtended angle by the small sphere. Then, \(\theta = 0 \rightarrow \alpha\)

\[
\Omega_0 = \int d\Omega f(\theta, \phi) = \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\alpha} d\theta \sin \theta = \int_{\phi=0}^{2\pi} d\phi (-\cos \alpha + 1) \\
\approx 2\pi \left( 1 - 1 + \frac{\alpha^2}{2} \right) \\
\approx \pi \alpha^2
\]

Using the small angle approximation we obtain

\[
\sin \alpha \approx \alpha = \frac{a}{r}
\]
and the expression for solid angle becomes
\[ \Omega_0 \approx \frac{\pi a^2}{r^2} \]

4. The gain function of the antenna is given as

\[ G(\theta, \phi) = \begin{cases} D & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ D \sin^2 \theta & \text{for } \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \]

Apply the relationship that specifies the solid angle integral over the gain function must be equal to a constant, i.e., \( \int d\Omega G(\theta, \phi) = 4\pi \), but in two separate regions of theta space because the gain function differs from 0 to \( \pi/2 \) and from \( \pi/2 \) to \( \pi \).

\[
2\pi D \left[ \int_{\theta=\pi/2}^{\pi} d\theta \sin \theta + \int_{\theta=\pi/2}^{\pi/2} d\theta (1 - \cos^2 \theta) \sin \theta \right] = 4\pi
\]

Next use “u” substitution in the integral on the right, let \( u = \cos \theta \), then \( du = -\sin \theta d\theta \)

\[
2\pi D \left[ \left( -\cos \theta \right)^{\pi/2} + \left( \cos \theta - \frac{\cos^3 \theta}{3} \right)^{\pi/2} \right] = 4\pi
\]

\[
2\pi D \left( 1 + \frac{2}{3} \right) = 4\pi
\]

\[
D = \frac{6}{5}
\]

5.

a) We have for the array a total electric field

\[ \vec{E} \propto e^{-jkr} \left[ e^{-jkd \cos \phi} + 2 + e^{jkd \cos \phi} \right] \]

\[ \propto \left( e^{-jkd \cos \phi/2} + e^{jkd \cos \phi/2} \right)^2 \]

\[ \propto \cos^2 \left( \frac{kd \cos \phi}{2} \right). \]

b) On the xy-plane along x-axis, we have \( \phi = 0^\circ \) or \( 180^\circ \), so \( \cos \phi = \pm 1 \). The null position satisfies

\[
\cos^2 \left( \frac{kd \cos \phi}{2} \right) = 0,
\]

\[
\frac{kd \cos \phi}{2} = m\pi + \frac{\pi}{2}
\]

\[
\frac{\pi d}{\lambda} = m\pi \pm \frac{\pi}{2}
\]

\[
d = (m \pm \frac{1}{2})\lambda,
\]

where \( m = 0, \pm 1, \pm 2, \cdots \).
6.

a) Now we have

$$\mathbf{E} \propto e^{-jkr} \left[ e^{j\psi} e^{-jkd \cos \phi} + 2 + e^{-j\psi} e^{jkd \cos \phi} \right]$$

$$\propto \left( e^{-j(kd \cos \phi - \psi)/2} + e^{j(kd \cos \phi - \psi)/2} \right)^2$$

$$\propto \cos^2 \left( \frac{kd \cos \phi - \psi}{2} \right).$$

b) To have field nulls along $\phi = 0^\circ$ and $\phi = 60^\circ$ we must have

$$kd - \angle \psi = (2m + 1)\pi,$$

$$\frac{kd}{2} - \angle \psi = (2n + 1)\pi$$

where $m$ and $n$ are integers which cannot be equal. Solve for $d$ and $\angle \psi$, we have

$$kd = 4(m - n)\pi,$$

$$d = 2(m - n)\lambda,$$

$$\angle \psi = (-4n + 2m - 1)\pi,$$

where $m, n = 0, \pm 1, \pm 2, \cdots$. There are infinitely many possible solutions. Shortest $d$ corresponds to having $m - n = 1$. 