1. The propagation of ocean waves on the surface of deep water (depth much longer than a wavelength \( \lambda = \frac{2\pi}{k} \)) is governed by the dispersion relation

\[
\omega = \sqrt{gk}
\]

where \( g \) represents gravitational acceleration and \( \omega, k \) are the frequency and wavenumber, respectively.

a) Make a sketch of \( \omega \) vs \( k \) over positive \( k \) for deep water ocean waves and based on the graph see whether you expect the group velocity \( v_g = \frac{\partial \omega}{\partial k} \) to be greater or less than the phase velocity \( v_p = \frac{\omega}{k} \).

b) Find the phase and group velocities \( v_p \) and \( v_g \) for deep water ocean waves. How are the two velocities related?

c) If a rock falling into a calm, deep ocean, excites a continuous spectrum of waves with periods longer than \( T_{\text{min}} \) seconds and shorter than \( T_{\text{max}} \) seconds, where \( T_{\text{max}} \gg T_{\text{min}} \), will waves oscillating with periods close to \( T_{\text{min}} \) or \( T_{\text{max}} \) reach a distant shore first after propagating away from the fallen rock? — Determine the answer in terms of \( \omega \) dependence of \( v_g \) and explain your reasoning.

2. For "bending waves" on solid rods (excited by applying an oscillatory force transverse to the rod) the phase velocity expression is known to be \( v_p = A\sqrt{\omega} \), where \( A > 0 \) is a real constant (which depends on density, radius, and Young’s modulus of the rod) and \( \omega \) is the oscillation frequency in rad/s.

a) Determine the unit for constant \( A \) in MKS system.

b) Make a sketch of \( \omega \) vs \( k \) over positive \( k \) and based on the graph see whether you expect the group velocity \( v_g = \frac{\partial \omega}{\partial k} \) to be greater or less than the phase velocity \( v_p = \frac{\omega}{k} \).

c) Determine the group velocity expression for bending waves.

d) Suppose on a very long rod a narrowband pulse of bending wave oscillations is excited, and the pulse is observed to travel on the rod from left to right. An observer moving to the right with the velocity of the pulse pays attention to the motion of the crests of the carrier of the pulse. Will he/she observe crest motions from right-to-left or from left-to-right? Explain clearly.

3. Using

\[
\omega_p = \sqrt{\frac{Ne^2}{me_o}}
\]

and the numerical values of electron mass and charge \( m \) and \( e \) in MKS:

a) Show that

\[
f_p = \frac{\omega_p}{2\pi} \approx \sqrt{80.6N} \approx 9\sqrt{N} \text{ Hz}
\]

if \( N \) is to be entered in elect/m^3 units.

b) Calculate \( f_p \) in MHz if \( N = 10^{12} \) elect/m^3.

4. A homogeneous collisionless plasma can be modeled as a perfect dielectric with \( \mu = \mu_o \) and a refractive index

\[
n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}
\]

where \( \omega_p \) is a real constant. A plane wave propagating in \( x \)-direction in such a plasma with a phasor \( \propto e^{-j\omega t} \) would have \( k = \frac{\omega}{c}n \) as usual.
a) What would be the “dispersion relation” in the plasma that explicitly states the wave frequency \( \omega \) as a function of wavenumber \( k \)?

b) How would the same dispersion relation be stated when the wavenumber \( k \) is expressed explicitly in terms of the wave frequency \( \omega \)?

c) What would be the phase velocity \( v_p \) and group velocity \( v_g \) in the plasma if \( \omega = \frac{5}{3} \omega_p \)?

5. Consider propagation of a plane TEM wave in a homogeneous collisionless plasma.

   a) If the electron density in the plasma is \( N = 10^{12} \) elect/m\(^3\) determine whether waves with the frequencies \( f_1 = \frac{\omega_1}{2\pi} = 6 \) MHz and \( f_2 = \frac{\omega_2}{2\pi} = 10 \) MHz are evanescent or propagating?

   b) For the evanescent wave identified in part (a) determine the distance \( d \) over which the field vector of the evanescent wave is reduced by a factor of \( \sqrt{e} \).

   c) How many decibels of reduction is the reduction of the wave amplitude by a factor of \( 1/e \)? Make sure you understand why you use the factor 10 or 20 in dB calculation — which one is the right one?? — and explain briefly why.

   d) For the propagating wave identified in part (a) determine the wavenumber \( k \), phase velocity \( v_p \), and wavelength \( \lambda \). How do these values compare with their counterparts in vacuum?

6. A monochromatic TEM wave of a frequency \( \omega = 2\pi f \) is incident on the ionosphere (ionized slab of the Earth’s upper atmosphere) from below. Modeling the ionosphere as a homogeneous perfect dielectric with a relative permeability of unity and relative permittivity

\[ \epsilon_r = 1 - \frac{\omega_p^2}{\omega^2} \]

answer the following questions.

   a) Express the wavenumber \( k \) of a TEM wave in the ionosphere in terms of the wave frequency \( \omega \), plasma frequency \( \omega_p \), and the speed of light \( c = \frac{1}{\sqrt{\mu_r \epsilon_r}} \).

   b) Over which range of frequencies \( \omega \) will the waves in the ionosphere be evanescent? Is \( k \) real or imaginary for evanescent waves?

   c) If a 9 MHz TEM wave is able to penetrate through the ionosphere as a propagating (rather than an evanescent) wave, what is the upper limit on the number density \( N \) of free electrons in the ionosphere per unit volume (in elect/m\(^3\) units)?

   d) What are the phase and group velocities \( v_p \) and \( v_g \) of the TEM wave in the ionosphere if \( N = 10^{11} \) elect/m\(^3\)?