1. The medium is dispersive, i.e., different frequency components will travel at different speeds, which is determined by the dispersion relation given as $\omega = \sqrt{gk}$.

   a) The gravitational acceleration constant $g$ is given as $9.822 \text{ m/s}^{-1}$, hence a general sketch of $\omega$ vs $k$ is shown in Figure 1.

   ![Figure 1](image-url)

   Figure 1 shows the solid-blue curve as the a sketch of the dispersion curve. At a particular operating point $(\omega_0, k_0)$, it is important to evaluate the slopes of a (1) the straight line from the origin to the operating point and (2) the slope of the dispersion curve at the operating point. Each one represents $v_p$ and $v_g$, respectively. Both are shown in the dashed curves in the figure.

   It is easy to see that the slope of the $v_p$ curve is greater and will be for all frequency points given the trajectory of the dispersion curve. Thus one can conclude that we expect the *group velocity* to be less than the *phase velocity* for ocean waves propagating on the surface of deep water.

   b)

   - Approach 1

   \[
   \begin{align*}
   \omega &= \sqrt{gk} \\
   \omega^2 &= gk \\
   \omega^2/k &= g \\
   \omega/k &= \frac{g}{\omega} \\
   \therefore \ v_p &= \frac{g}{\omega}
   \end{align*}
   \]
\[ \omega = \sqrt{gk} \rightarrow \omega^2 = gk \]
\[ \frac{\partial}{\partial k} (\omega^2 = gk) = 2\omega \frac{\partial \omega}{\partial k} = g \]
\[ \therefore \ v_g = \frac{\partial \omega}{\partial k} = \frac{g}{2\omega} \]

- **Approach 2**

\[ \omega = \sqrt{gk}, \]

then with \( k = \frac{\omega}{v_p} \)

\[ \omega = \sqrt{\frac{g}{v_p}} \omega \]
\[ \omega^2 = g \frac{\omega}{v_p} \]
\[ \therefore \ v_p = \frac{g}{\omega} \]
\[ \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k} \sqrt{gk} \]
\[ = \frac{1}{2} (gk)^{-1/2} \cdot g \]
\[ = \frac{g}{2\sqrt{gk}}. \]

With \( k = \frac{\omega^2}{g} \),

\[ v_g = \frac{\partial \omega}{\partial k} = \frac{g}{2\omega} \]

- Relationship: \( v_g = \frac{1}{2} v_p \) (as predicted), also note how both decreases as frequency increases.

c) The rock falling into the calm, deep ocean creates a spectrum of waves with periods longer than \( T_{min} \) and shorter than \( T_{max} \), i.e., \( T_{min} \leq T_i \leq T_{max} \), where \( T_i \) exemplifies the period of an individual frequency component.

Since the period is inverse proportional to frequency, i.e., \( T \propto \frac{1}{\omega} \), equate \( T_{min} \rightarrow \omega_{max} \) and \( T_{max} \rightarrow \omega_{min} \). This essentially indicates some type of bandwidth about the center frequency with \( \omega_{min} \leq \omega_i \leq \omega_{max} \).

Dropping of the rock produces waves of a wide spectrum of frequencies propagating toward the shore. If we were dealing with just a single frequency component (e.g., a single sinusoid), we would utilize the phase velocity. Since this is not the case, utilize the group velocities for several groups of waves having adjacent frequency bands within the excited spectrum.

At the extremes, the group of waves with frequencies near \( \frac{1}{T_{min}} \) travel at one \( v_g \) while the
group of those near \( \frac{1}{T_{\text{min}}} \) travel at another \( v_g \)—both dictated by the dispersion relationship and the frequency dependence of \( v_g \).

Therefore, if the shore was located at some distance \( r_o \) from the drop location, then the time to traverse the distance to the shore is given by

\[
T_{\text{travel}} = \frac{r_o}{v_g}
\]

- \( T_{\text{travel}} \bigg|_{T_{\text{min}}} = \frac{r_0}{g/2\omega_{\text{max}}} = \frac{2\omega_{\text{max}}}{g} \)
- \( T_{\text{travel}} \bigg|_{T_{\text{max}}} = \frac{r_0}{g/2\omega_{\text{min}}} = \frac{2\omega_{\text{min}}}{g} \)

Thus,

\[
\frac{T_{\text{travel}}}{T_{\text{min}}} \bigg|_{T_{\text{max}}} = \frac{\omega_{\text{max}}}{\omega_{\text{min}}} > 1
\]

which implies that \( T_{\text{travel}} \bigg|_{T_{\text{min}}} > T_{\text{travel}} \bigg|_{T_{\text{max}}} \), i.e., it takes a **longer time** for the **shorter period wave** \( (T_{\text{min}}) \), to reach the shore.

For this type of dispersion, **lower** frequency components travel with a faster \( v_g \) and higher frequency components travel with a slower \( v_g \). This is due to a combination of the reasons discussed above, but mainly due to the inverse relationship to frequency in the dispersion relationship.

- **Physical example**

  Imagine yourself sitting in a small boat (e.g., canoe) docked along the shore. Assume a disturbance in the ocean is created some distance away (e.g., a large ship creating a wake) and the waves propagate toward you. When the waves first hit you, the boat will rock with a low frequency oscillation. This is due to the lower frequency components of the wave reaching the boat first. The intensity (frequency) of the oscillations will then increase as the higher frequency components take up the back end of the wave.

2. In this case, the phase velocity is given: \( v_p = A\sqrt{\omega} \)

   a) \( v_p = A\sqrt{\omega} \) implies \( A = \frac{v_p}{\sqrt{\omega}} \), then a unit analysis yields

   \[
   A \rightarrow \frac{\text{m/s}}{\sqrt{\text{rad/s}}} = \frac{\text{m}}{\sqrt{\text{rad}\sqrt{s}}} = \frac{\text{m\sqrt{rad}\sqrt{s}}}{\text{rad} \cdot \text{s}}
   \]

   b) To sketch the plot, keep in mind \( A \) is just a constant that is greater than 0 and thus it will not effect predicting the relative phase and group velocities. But before a comparison can be made,
the dispersion relationship is needed. Write the dispersion relationship either as \( \omega \rightarrow f(k) \) or \( k \rightarrow f(\omega) \).

\[
v_p = A\sqrt{\omega} = \frac{\omega}{k}
\]

then

\[
A\sqrt{\omega} = \frac{\omega}{k}
\]

\[
k^2 = \frac{\omega}{A^2}
\]

or

\[
\omega = A^2k^2
\]

(1)

Now, follow the same approach as in Problem #1, part a). Figure 2 displays the sketch.

![Figure 2](image)

Using the same analysis as before one can see that at the operating point, it is expected that the group velocity will be greater than the phase velocity for “bending” waves propagating on a solid rod.

c) To find the \( v_g \) use the dispersion relation already determined in (1), then

\[
v_g = \frac{\partial \omega}{\partial k}
\]

\[
= \frac{1}{\partial k}
\]

\[
= \frac{1}{\partial \omega} \left( \sqrt{\frac{\omega}{A^2}} \right)
\]

\[
\therefore v_g = 2A\sqrt{\omega}
\]
• Note how \( v_g = 2 \cdot v_p \) (as predicted).

d) Answer: **right-to-left**, a number of points are in order.

• Since this is a narrow band signal acting on the rod, the pulse of bending waves (i.e., the packets) are traveling at the group velocity from **left-to-right** along the rod. This is the speed at which the pulse, the information, is traveling—not at the phase velocity.

• The observer is also moving **to the right** with the velocity of the pulse traveling at the **group velocity**.

• Recall from the lecture notes, for a sufficiently narrow band pulse, each frequency component takes on a co-sinusoid form. The superposition waveform in general can be described in terms of an envelope and a carrier. The envelope travels at the **group velocity**, while the carrier travels at the **phase velocity**. Again, \( v_g > v_p \) and the signal is narrowband.

• Finally, since the observer is eyeing a point on the **carrier’s crest**, moving at \( v_p \), which is slower than \( v_g \), it appears as if that point on the crest is **advancing** toward the observer, moving **right-to-left**, traveling backwards.

3. In a homogeneous collisionless plasma, the plasma frequency is estimated as

\[
\omega_p = \sqrt{\frac{Ne^2}{me_0}} \tag{2}
\]

a) Given the correct numerical values for the electron mass and charge in MKS, in (2)

\[
\omega_p \approx \sqrt{\frac{(N \text{elect/m}^3) \left(1.602 \times 10^{-19} \text{ C/elec} \right)^2}{(9.109 \times 10^{-31} \text{ kg}) 8.854 \times 10^{-12} \text{ C}^2/(\text{kg m/s}^2 \cdot \text{m}^2)}}
\]

Then,

\[
2\pi f_p = \sqrt{3186 N} \quad f_p = \sqrt{\frac{3186 N}{(2\pi)^2}} \quad f_p \approx 9\sqrt{N} \text{ Hz} \tag{3}
\]

Note how the calculation is dimensionally correct via the units.

b) If the electron density is \( N = 10^{12} \text{ elect/m}^3 \), then in (3)

\[
f_p = 9\sqrt{10^{12}} \text{ Hz} = 9 \times 10^6 \text{ Hz}
\]

\[
f_p = 9 \text{ MHz}
\]
a) Given the index of refraction $n$ as
\[ n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}, \]  
then
\[ k = \frac{\omega}{cn} \]
\[ k = \frac{\omega}{c} \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \]
\[ k^2 = \omega^2 - \frac{\omega_p^2}{c^2} \]
\[ \omega^2 = c^2 k^2 + \omega_p^2 \]
\[ \omega = \sqrt{c^2 k^2 + \omega_p^2} \]  
(5)

b) From (5), it can easily be written as
\[ k^2 = \frac{\omega^2 - \omega_p^2}{c^2} \]
\[ k = \frac{1}{c} \sqrt{\omega^2 - \omega_p^2} \]

c)
\[ v_p = \frac{\omega}{k} \]
\[ = \frac{\omega}{c} \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \]
\[ = \frac{\omega}{c} \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \]
\[ = \sqrt{1 - \frac{\omega_p^2}{(\omega_p/c)^2}} \]
\[ = \frac{5}{4} c = 3.75 \times 10^8 \text{ m/s} \]

Note how $v_p$ is greater than the speed of light $c$.

\[ v_g = \frac{\partial \omega}{\partial k} \]
\[ = \frac{\partial}{\partial k} \left( \sqrt{c^2 k^2 + \omega_p^2} \right) \]
\[ = \frac{k c^2}{\sqrt{c^2 k^2 + \omega_p^2}} \]
and since $k = \frac{1}{c} \sqrt{\omega^2 - \omega_p^2}$ and from (5) $k^2 c^2 = \omega^2 - \omega_p^2$ then

$$v_g = \frac{1}{c} \sqrt{\omega^2 - \omega_p^2} \cdot \frac{c^2}{\sqrt{\omega^2 - \omega_p^2 + \omega_p^2}}$$

$$= c \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

$$= c \sqrt{1 - \frac{\omega^2}{(5/3\omega_p)^2}}$$

$$= \frac{4}{5} c = 2.4 \times 10^8 \text{ m/s}$$

Note in this case, $v_g$ is less than the speed of light $c$, as it should be due to the laws of physics! Also note the $v_p \cdot v_g = c^2$ relationship holds for waves in a homogeneous collisionless plasma.

5.

a) Utilize the plasma frequency $\omega_p$ to determine the propagation status of the waves in the plasma. Since electron density is again given as $N = 10^{12} \text{ elect/m}^3$, then in (3)

$$f_p = 9 \text{ MHz}.$$  

Instinctively, one should know that when $f_0 > f_p$ the wave propagates in the plasma and when $f_0 < f_p$ the wave is evanescent, i.e., it’s amplitude attenuates as it travels in the medium. This result is due to the index of refraction given in (4) for a homogeneous collisionless plasma. In the former case, $n$ is purely real and thus the $k \cdot r$ term in $e^{-jk \cdot r}$ will allow for propagation. In the latter case, $n$ is purely imaginary and thus the $k \cdot r$ term becomes imaginary, making $e^{-jk \cdot r}$ an amplitude attenuating factor.

For the frequencies of interest:

- $f_1 = 6 \text{ MHz} < f_p$ \implies evanescent
- $f_2 = 10 \text{ MHz} > f_p$ \implies propagating

b) The distance $d$ over which the wave amplitude of the evanescent wave is reduced by a factor $1/e$ is indicated by the skin depth, $\delta$. Recall, $\delta = \frac{1}{|k|}$ and for the evanescent wave

$$|k| = \left| \frac{\omega}{c} \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \right|$$

$$= \left| \pm \frac{\omega}{c} \sqrt{\frac{\omega_p^2}{\omega^2} - 1} \right|.$$

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Then

\[
\begin{align*}
d &= \delta = \frac{1}{\frac{\omega}{c} \sqrt{\frac{\omega_p^2}{\omega^2} - 1}} \\
&= \frac{1}{\frac{2\pi (6 \times 10^6)}{3 \times 10^8} \sqrt{\frac{(9 \text{ MHz})^2}{(6 \text{ MHz})^2} - 1}} \\
d &= 7.118 \text{ m}
\end{align*}
\]

c) Since the wave amplitude is reduced by a factor $1/e$ then in dB, this amounts to

\[
20 \log_{10} \left( \frac{1}{e} \right) = -8.686 \text{ dB}
\]

$20 \log_{10}(\cdot)$ is used in this case because we are concerned with the field amplitude and not power. If power was a concern, then one would use $10 \log_{10}(\cdot)$. Both are equivalent since power $\propto$ field$^2$.

d) For the propagating wave:

\[
k = \frac{\omega}{c} \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \\
= \frac{2\pi (10 \times 10^6)}{3 \times 10^8} \sqrt{1 - \frac{(9 \text{ MHz})^2}{(10 \text{ MHz})^2}} \\
= 0.0913 \text{ rad/m}
\]

Compare to free space $k$,

\[
k_0 = \frac{\omega}{c} \\
= \frac{2\pi (10 \times 10^6)}{3 \times 10^8} \\
= 0.2094 \text{ rad/m}
\]

\[
v_p = \frac{\omega}{k} \\
= \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \\
= \frac{3 \times 10^8}{\sqrt{1 - \frac{(9 \text{ MHz})^2}{(10 \text{ MHz})^2}}} \\
= 6.88 \times 10^8 \text{ m/s}
\]

Compare to the free space $v_p$, 

\[
c = 3.0 \times 10^8 \text{ m/s}
\]
\[
\lambda = \frac{2\pi}{k} = \frac{v_p}{f} = \frac{2\pi}{0.0913 \text{rad/m}} = 68.8 \text{ m}
\]

Compare to free space \(\lambda\),
\[
\lambda_0 = \frac{c}{f} = \frac{3.0 \times 10^8}{10 \times 10^6} = 30 \text{ m}
\]

Compared to the free space counterparts, the results dictate the wavenumber \(k\) is reduced in the plasma, the phase velocity \(v_p\) is more than doubled, and the wavelength \(\lambda\) is more than doubled in the plasma. All the plasma results are scaled versions of their free space counterparts, scaled by the index of refraction \(n\).

6.

a) From Problem 2(b), we have
\[
k = \frac{1}{c} \sqrt{\omega^2 - \omega_p^2}.
\]

b) For evanescent waves, \(\omega < \omega_p\). As a result, \(k\) is imaginary.

c) For propagating waves, we have \(f > f_p = 9\sqrt{N}\), therefore,
\[
N < \frac{f^2}{81} = \frac{(9 \times 10^6)^2}{81} = 10^{12} \text{elect/m}^3.
\]

d) From (c), we have
\[
f = 9 \times 10^6 \text{Hz},
\]
\[
f_p = 9\sqrt{N} = 2.846 \times 10^6 \text{Hz}
\]
so
\[
\frac{f_p^2}{f^2} = 0.1
\]

From Problem 2(c), we have the phase velocity
\[
v_p = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} = \frac{c}{\sqrt{1 - \frac{f_p^2}{f^2}}} = \frac{3 \times 10^8}{\sqrt{0.9}} = 3.1623 \times 10^8 \text{m/s}.
\]

For the group velocity,
\[
v_g = c \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = c \sqrt{1 - \frac{f_p^2}{f^2}} = 3 \times 10^8 \sqrt{0.9} = 2.8460 \times 10^8 \text{m/s}.
\]