**Polarization**

\[
\vec{F}_1 = \hat{x} f_1 \cos(\omega t + \phi) \\
\vec{F}_2 = \hat{y} f_2 \cos(\omega t + \phi)
\]

\[
\alpha = \tan^{-1} \left( \frac{F_y}{F_x} \right) = \tan^{-1} \left( \frac{f_2}{f_1} \right)
\]

\(\alpha\) is constant; we have linear polarization

Suppose now that:

\[
\vec{F}_1 = \hat{x} f_1 \cos(\omega t + \phi) \\
\vec{F}_2 = \hat{y} f_2 \cos(\omega t + \phi - \pi/2) = \hat{y} \sin(\omega t + \phi)
\]

then,

\[
\alpha = \tan^{-1} \left( \frac{F_y}{F_x} \right) = \tan^{-1} \left[ \tan(\omega t + \phi) \right] = \omega t + \phi
\]

\(\alpha\) is proportional to time \(\Rightarrow\) circular polarization. Tip of vector \(\vec{F} = \vec{F}_1 + \vec{F}_2\) draws a circle

In general,

\[
\vec{E}(t) = \hat{x} E_x(t) + \hat{y} E_y(t) = \hat{x} a \cos(\omega t + \alpha_x) + \hat{y} a \cos(\omega t + \alpha_y)
\]

with

\[
-\frac{\pi}{2} \leq \alpha_x, \alpha_y \leq \frac{\pi}{2}
\]

In phasor form, \(\vec{E} = \hat{x} E_x + \hat{y} E_y\) where \(E_x\) and \(E_y\) are now phasors. Define two elementary complex vectors

\[
\vec{L} = \hat{x} + j \hat{y} \quad \vec{R} = \hat{x} - j \hat{y}
\]

\[
\vec{L} \cdot \vec{L} = 0 = \vec{R} \cdot \vec{R} \quad \vec{L} = \vec{R}^* \\
\vec{L} \cdot \vec{L}^* = \vec{R} \cdot \vec{R}^* = 2 
\]

Not unit vectors
Also,
\[ \hat{x} = \frac{L + R}{2} \quad \text{and} \quad \hat{y} = \frac{L - R}{2j} \]

\[ \tilde{L}(t) = \hat{x} \cos \omega t - \hat{y} \sin \omega t \]
when \( \omega t = 0 \), \( \tilde{L}(0) = \hat{x} \)
when \( \omega t = \frac{\pi}{2} \), \( \tilde{L}\left(\frac{\pi}{2}\right) = -\hat{y} \)

moves clockwise with time

Similarly,
\[ \tilde{R}(t) = \hat{x} \cos \omega t + \hat{y} \sin \omega t \]
when \( \omega t = 0 \), \( \tilde{R}(0) = \hat{x} \)
when \( \omega t = \frac{\pi}{2} \), \( \tilde{R}\left(\frac{\pi}{2}\right) = +\hat{y} \)

moves in counterclockwise direction

We then get
\[ \tilde{E} = \hat{x}E_x + \hat{y}E_y = \frac{1}{2}\left(E_x - jE_y\right)\tilde{L} + \frac{1}{2}\left(E_x + jE_y\right)\tilde{R} = E_L\tilde{L} + E_R\tilde{R} \]

shows that linear and circular polarizations are related
\[ E_L = \frac{\left(E_x - jE_y\right)}{2} \quad E_R = \frac{\left(E_x + jE_y\right)}{2} \]
\[ E_x = E_L + E_R \quad E_y = j(E_L - E_R) \]

An arbitrary polarization can be expressed as a combination of RHP and LHP waves by means of \( \tilde{R} \) and \( \tilde{L} \) by decomposing polarized complex vector in terms of its R and L components
\[ \tilde{E}_L(t) = |E_L|\left\{\hat{x} \cos(\omega t + \theta_L) + \hat{y} \cos(\omega t + \theta_L + \pi / 2)\right\} \]
\[ \tilde{E}_R(t) = |E_R|\left\{\hat{x} \cos(\omega t + \theta_R) + \hat{y} \cos(\omega t + \theta_R - \pi / 2)\right\} \]

where \( E_L = |E_L|e^{j\phi_L} \) and \( E_R = |E_R|e^{j\phi_R} \)

Total field \( \tilde{E}(t) \) at any time is vectorial sum of \( \tilde{E}_R(t) \) and \( \tilde{E}_L(t) \)
Need to study the shape and inclination of ellipse. Define
\[
    \Gamma = \frac{E_L}{E_R} \text{ if } |E_L| \leq |E_R|
\]
\[
    \Gamma = \frac{E_R}{E_L} \text{ if } |E_R| \leq |E_L|
\]

For simplicity, assume \(|E_L| \leq |E_R|\) then
\[
    \tilde{E} = E_R \left( \tilde{R} + \Gamma \tilde{L} \right) = |E_R| e^{i\theta_0} \left( \tilde{R} + \Gamma \tilde{L} \right)
\]

Shape of ellipse is determined by \( \tilde{R} + \Gamma \tilde{L} \)

Axis ratio \( AR = \frac{A}{B} = \frac{|E_R| + |E_L|}{|E_R| - |E_L|} \)