WAVEGUIDES

Maxwell's Equation

\[ \nabla^2 \mathbf{E} + \omega^2 \mu \varepsilon \mathbf{E} = 0 \]  \hspace{1cm} (A)

\[ \nabla^2 \mathbf{H} + \omega^2 \mu \varepsilon \mathbf{H} = 0 \]  \hspace{1cm} (B)

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set \( E_z = 0 \). This defines the TE modes.

From \( \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \), we have

\[ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \Rightarrow +j\beta_z E_y = -j\omega \mu H_x \]  \hspace{1cm} (1)

\[ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \Rightarrow -j\beta_z E_x = -j\omega \mu H_y \]  \hspace{1cm} (2)

\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega \mu H_z \]  \hspace{1cm} (3)

From \( \nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} \), we get

\[
\begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
H_x & H_y & H_z
\end{vmatrix}
\]

\[ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega \varepsilon E_x \Rightarrow \frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega \varepsilon E_x \]  \hspace{1cm} (4)
\[
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j \omega \varepsilon E_y \Rightarrow -j \beta_z H_x - \frac{\partial H_z}{\partial x} = j \omega \varepsilon E_y
\]  \hspace{1cm} (5)

\[
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0
\]  \hspace{1cm} (6)

We want to express all quantities in terms of \( H_z \).

From (2), we have

\[
H_y = \frac{\beta_z E_x}{\omega \mu}
\]  \hspace{1cm} (7)

in (4)

\[
\frac{\partial H_z}{\partial y} + j \beta_z^2 \frac{E_x}{\omega \mu} = j \omega \varepsilon E_x
\]  \hspace{1cm} (8)

Solving for \( E_x \)

\[
E_x = \frac{j \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial y}
\]  \hspace{1cm} (9)

From (1)

\[
H_x = \frac{-\beta_z E_y}{\omega \mu}
\]  \hspace{1cm} (10)

in (5)

\[
+j \frac{\beta_z^2 E_y}{\omega \mu} - \frac{\partial H_z}{\partial x} = j \omega \varepsilon E_y
\]  \hspace{1cm} (11)

so that

\[
E_y = \frac{-j \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial x}
\]  \hspace{1cm} (12)

\[
H_x = \frac{j \beta_z}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial x}
\]  \hspace{1cm} (13)

\[
H_y = \frac{j \beta_z}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial y}
\]  \hspace{1cm} (14)
Ez = 0 \tag{15}

Combining solutions for Ex and Ey into (3) gives

\[
\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] H_x \tag{16}
\]

**RECTANGULAR WAVEGUIDES**

If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

**TE Modes**

The general solution for TE modes with Ez=0 is obtained from (16)

\[
H_x = e^{-j\beta_x z} \left[ Ae^{-j\beta_y x} + Be^{+j\beta_y x} \right] \left[ Ce^{-j\beta_y y} + De^{+j\beta_y y} \right] \tag{17}
\]

\[
E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} e^{-j\beta_x z} \left[ Ae^{-j\beta_y x} + Be^{+j\beta_y x} \right] \left[ Ce^{-j\beta_y y} + De^{+j\beta_y y} \right] \tag{18}
\]

\[
E_x = \frac{j\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} e^{-j\beta_x z} \left[ Ae^{-j\beta_y x} + Be^{+j\beta_y x} \right] \left[ Ce^{-j\beta_y y} + De^{+j\beta_y y} \right] \tag{19}
\]

At y=0, Ex=0 which leads to C=D

At x=0 Ey=0 which leads to A=B

\[
H_x = H_o e^{-j\beta_z z} \cos \beta_x x \cos \beta_y y \tag{20}
\]

\[
E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_o e^{-j\beta_z z} \sin \beta_x x \cos \beta_y y \tag{21}
\]
\[ E_x = \frac{j\beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z} \cos\beta_x x \sin\beta_y y \]  \hspace{1cm} (22)

At \( x=a \), \( E_y=0 \); this leads to \( \beta_x = \frac{m\pi}{a} \)

At \( y=b \), \( E_x=0 \); this leads to \( \beta_y = \frac{n\pi}{b} \)

The dispersion relation is obtained by placing (20) in (16)

\[ \beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2 \mu \varepsilon \]  \hspace{1cm} (23)

\[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \beta_z^2 = \omega^2 \mu \varepsilon \]  \hspace{1cm} (24)

and

\[ \beta_z = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]  \hspace{1cm} (25)

The guidance condition is

\[ \omega^2 \mu \varepsilon > \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \]  \hspace{1cm} (26)

or

\( f > f_c \) where \( f_c \) is the cutoff frequency of the \( \text{TE}_{mn} \) mode given by the relation

\[ f_c = \frac{1}{2\sqrt{\mu \varepsilon}} \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2} \]  \hspace{1cm} (27)

The \( \text{TE}_{mn} \) mode will not propagate unless \( f \) is greater than \( f_c \). Obviously, different modes will have different cutoff frequencies.

**TM Modes**

The transverse magnetic modes for a general waveguide are obtained by assuming \( H_z =0 \). By duality with the TE modes, we have
\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = [\beta_z^2 - \omega^2 \mu \varepsilon] E_z
\]  \hfill (28)

with general solution

\[
E_z = e^{-j\beta_z z} \left[ Ae^{-j\beta_z x} + Be^{j\beta_z x} + Ce^{-j\beta_z y} + De^{j\beta_z y} \right]
\]  \hfill (29)

The boundary conditions are

At \(x=0\), \(E_z=0\) which leads to \(A=-B\)

At \(y=0\), \(E_z=0\) which leads to \(C=-D\)

At \(x=a\), \(E_z=0\) which leads to \(\beta_x = \frac{m\pi}{a}\)

At \(y=b\), \(E_z=0\) which leads to \(\beta_y = \frac{n\pi}{b}\)

so that the generating equation for the \(\text{TM}_{mn}\) modes is

\[
E_z = E_0 e^{-j\beta_z z} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]  \hfill (30)

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR \(\text{TE}\) AND \(\text{TM}\) MODES.

For additional information on the field equations see Rao (6th Edition), page 607 Table 9.1.

There is no \(\text{TE}_{00}\) mode

There are no \(\text{TM}_{m0}\) or \(\text{TM}_{0n}\) modes

The first \(\text{TE}\) mode is the \(\text{TE}_{10}\) mode

The first \(\text{TM}\) mode is the \(\text{TM}_{11}\) mode

**Impedance of a Waveguide**

For a \(\text{TE}\) mode, we define the transverse impedance as
\[ \eta_{gTE} = \frac{-E_y}{H_x} = \frac{E_x}{H_y} = \frac{\omega \mu}{\beta_z} \]

From the relationship for \( \beta_z \) and using
\[ f_c^2 = \frac{1}{4 \pi^2 \mu \varepsilon} \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right] \] (31)

we get
\[ \eta_{gTE} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}} \] (32)

where \( \eta \) is the intrinsic impedance \( \eta = \sqrt{\frac{\mu}{\varepsilon}} \). Analogously, for TM modes, it can be shown that
\[ \eta_{gTM} = \eta \sqrt{1 - \frac{f_c^2}{f^2}} \] (33)

**Power Flow in a Rectangular Waveguide (TE_{10})**

The time-average Poynting vector for the TE_{10} mode in a rectangular waveguide is given by
\[ \langle P \rangle = \frac{1}{2} \Re \{ \mathbf{E} \times \mathbf{H}^* \} = \hat{z} \frac{|E_0|^2}{2} \frac{\beta_z a}{\omega \mu} \sin^{2} \frac{\pi x}{a} \] (34)

\[ \langle \text{Power} \rangle = \int_{0}^{a} \int_{0}^{b} \frac{|E_0|^2}{2} \frac{\beta_z a}{\omega \mu} \sin^{2} \frac{\pi x}{a} \, dx \, dy \] (35)

\[ \langle \text{Power} \rangle = \frac{|E_0|^2}{4} \frac{\beta_z a b}{\omega \mu} = \frac{|E_0|^2}{4 \eta_{gTE_{10}}} \] (36)

Therefore the time-average power flow in a waveguide is proportional to its cross-section area.